Strategies and Thinking about Number in Children Aged 9-11 Years

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This report is a review of the international literature on number knowledge, number strategies, and frameworks for classifying children's (aged 9-11) learning of number. Research into children's thinking in these areas suggests a general progression from concrete thinking tied to physical models and counting methods, to abstract thinking using known number facts and relationships. Using information about children's thinking in teacher professional development has resulted in increased effectiveness in teaching. Although the suggested progressions are not developmental stages in a formal sense, they can guide instructional sequences and help teachers to decide 'where to next' for a child.

Table of Contents

Past mathematics curricula for the primary-aged child have been based on an analysis of the steps provided by the structure of mathematics. Current reform in mathematics education emphasises the role of children's thinking in their learning of mathematics, and proposes a curriculum based on the development of children's concepts.

The research summarised in this review suggests that there are identifiable progressions in children's thinking in the number domain of mathematics, and that these have an impact on the way that children solve problems, and they way that they learn more advanced mathematics. This review will outline and compare research findings in this area, fitting these into an overall pattern suggested by the research literature. This review focuses on children in the upper primary school, of ages 9 to 11.

The ways in which children think about numbers and operate on these numbers are often referred to as 'strategies'. This term is used widely in the literature, and requires further examination. Within the domain of mathematics education, terms such as skill, understanding, knowledge, concept, procedure, process and strategy are used in different ways by different authors. The types of cognitive behaviour discussed by authors in the field of children's strategic thinking in number require both a sense of what numbers are and what can be done with numbers. A mental addition method may require an understanding of place value, the additive composition of numbers and the process of addition in order to be used successfully. The understanding of the numbers and the skill of adding them are bound together in the process of solving the problem – neither makes sense without the other. For this reason, this review considers research into children's number sense, the development of place value understandings and the acquisition of the ability to operate on numbers. A major intersection between number sense and the ability to operate on numbers is in mental arithmetic strategies, which are a key focus of mathematics education reform Beishuizen, 1999; McIntosh, 1996).

The importance of number sense in defining strategies

A framework which presents number sense as the backbone of the number domain is proposed by McIntosh, Reys & Reys (1992). Defining number sense as "a propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information ... (resulting) … in an expectation that numbers are useful and that mathematics has a certain regularity" (p. 4). They propose three strands to number sense: (1) a knowledge of and facility with numbers, (2) knowledge of and facility with operations and (3) applying knowledge of and facility with numbers and operations to computational settings. This definition encompasses all the behaviour defined by other authors as strategy use, including the metacognitive skills of strategy selection and flexibility. Thus, strategies are seen as embedded within number sense, perhaps arising from it as they increase in sophistication.

Procedures and concepts in defining strategies

Within this broad definition there remain issues about the nature of strategies. Strategies for solving particular problem types are often presented as procedures that are followed in response to the stimulus problem. Beishuizen (1999) notes that there is a perception that procedures are in some way inferior, suggesting rote learning or unthinking application of a taught method. For Beishuizen (1999), strategy is the "choice out of options related to problem structure" and procedure is "the execution of computational steps related to the numbers in the problem" (p. 127). Children's problem solving attempts are coded in terms of this distinction. At times the strategy and procedure are in correspondence, at times there is a 'misfit' of strategy and procedure, resulting in errors. This definition of strategy still fits within the McIntosh, Reys & Reys (1992) definition, in the third strand of 'application', but is narrower than other definitions that include Beishuizen (1999) 'procedures' as strategies (for example, counting on in tens). Fuson & Smith (1999) use 'method', to avoid the implications of thoughtfulness in 'strategy' and the implications of rote learning in 'procedures'.

Other authors look to new terms to help define children's thinking. Gray & Tall (1994) propose the term 'procept', to embody the idea that mathematical symbols and expressions can contain a process and a concept. For example, simple addition statements (5+4) contain the process of addition (for example, start at 5 and count on 4) and the concept of a sum (5 and 4 is 9). The symbol (numeral or other notation) comes to represent the range of understandings the child collects about numbers; this is 'proceptual thinking' (p. 122). Gray &

Tall (1994) use this term to examine the categories of strategies discussed by authors such as Carpenter & Moser (1983) (See Figure 2), showing that more sophisticated strategies build more powerful procepts in children's minds. Essentially, Gray and Tall (1994) are discussing the building of increasingly useful mathematical structures that contain both process and concept in such as way that the processes do not become isolated from conceptual sense. Their definition thus extends the McIntosh, Reys & Reys (1992) framework, combining the ideas about number and the ideas about operations into more powerful units. Gray & Tall (1994) suggest that this is the way mathematicians work, and reflects their 'observed cognitive reality' (p. 121).

Strategies defined

For the purposes of this review 'strategies' are given a broad definition, in line with the number sense definition of McIntosh, Reys & Reys (1992), in order to cover as much of the relevant literature as possible. It will be seen that each sector of the research literature, while developed separately in the main, has important interrelationships which are supported by this definition. Many of the studies reviewed here are interested in children's thinking processes, and thus focus on mental arithmetic. The link between thinking and recording in mathematics is explored below.

Theoretical underpinnings

The notion of children's thinking being an important factor in determining teaching and learning has its origins in the work of Piaget and Vygotsky. Piaget's theories on fundamental mathematical concepts (Piaget, 1952) have been important precursors to modern examination of children's thinking. The basic Piagetian notions of a step-wise developmental sequence with mutually exclusive stages underlies some of the proposed frameworks in the strategy literature (Steffe, Cobb & von Glaserfeld, 1988). Other frameworks emphasise flexibility and the possible non-hierarchical nature of some strategies (Bills & Gray, 2001, Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema, 1997). In addition, variations of Piaget's clinical interview technique are the most common methodology used in this field (for example: Anghileri, 1989; Clark & Kamii, 1996, Mulligan & Mitchelmore, 1997, Young-Loveridge, 2000). Vygotsky's (1978) 'zone of proximal development' provides the momentum for the teacher change aspect to several of the research programmes (Steffe et al 1988, Thomas & Ward, 2001). If teachers can identify a child's strategy, and then can identify the next step for the child, teaching should be more effective. This has been shown to be true across a range of programmes (Fennema, Carpenter, Levi, Jacobs & Empson, 1996; Jones, Thornton, Putt, Hill, Mogill, Rich & Van Zoest, 1996; Fuson, Smith & LoCicero, 1997; Thomas & Ward, 2001).

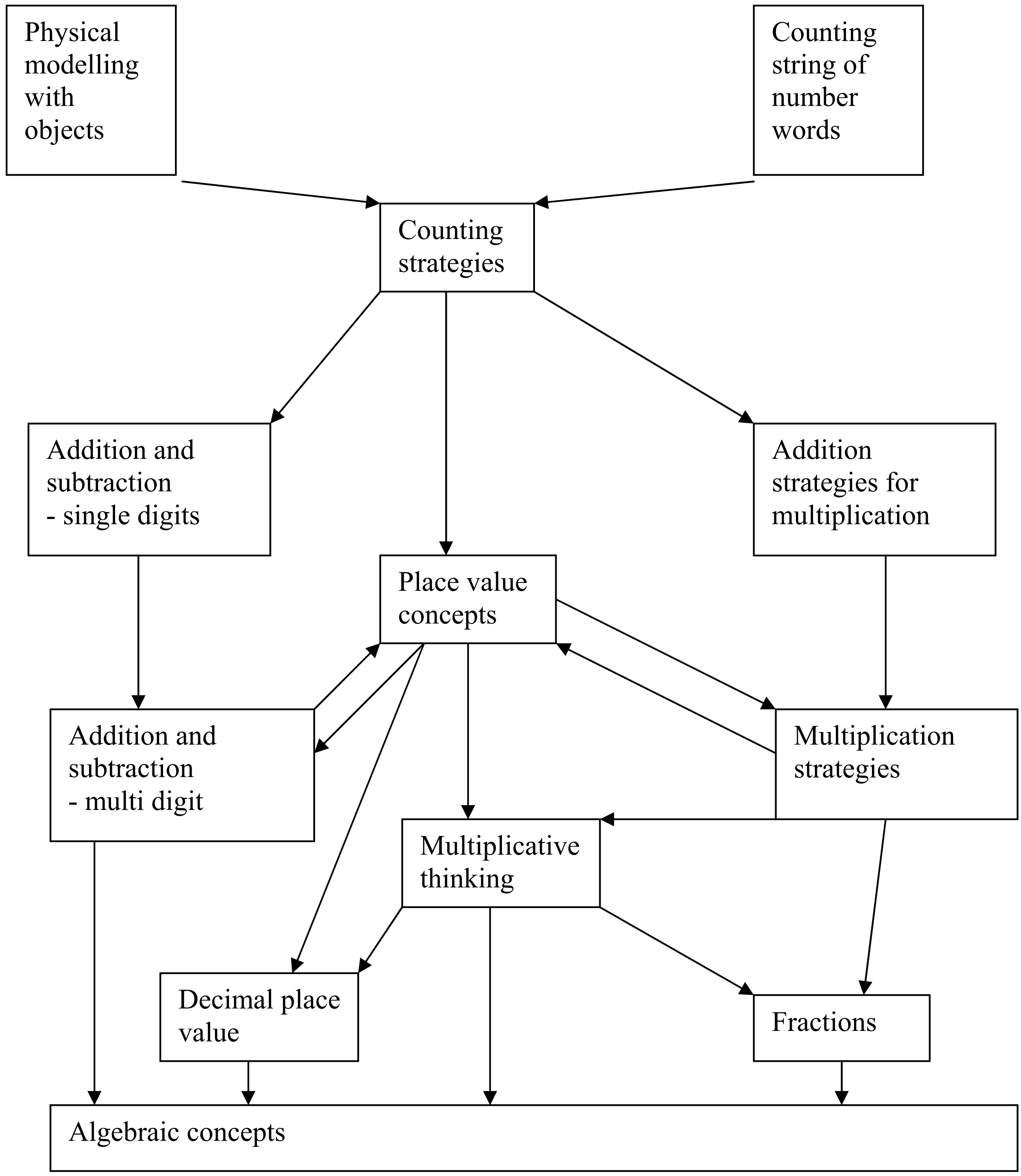
Emerging trends

Research into children's strategies falls into areas determined by the part of mathematics they investigate. Researchers only occasionally link these areas together (for example: McIntosh, Reys & Reys, 1992; Peters, 1997). The relevant areas are addition and subtraction (single digit and multi-digit often considered separately), place value, number sense, multiplication and division and fractions and decimals. These areas are linked, however, in two ways. Firstly, development within each area seems to follow a similar path, from the concrete to the abstract. Problem solution in each area begins with modelling, progresses to counting strategies of increasing sophistication and finishes with known ‘items’ of mathematics, for example basic facts. This trend is outlined more fully below, and in particular in Tables 1, 2 and 3.

Secondly, they form a pathway themselves, increasing in abstraction and mathematical power. This is shown in Figure One. These areas are more interconnected than this diagram implies, but it shows the key relationships in terms of the strategy literature. Each ‘set’ of strategies is building towards a qualitatively different understanding. Counting strategies will help you to solve addition and multiplication problems, but they provide too much of a cognitive load to be helpful for mentally multiplying larger numbers (Beishuizen, 1999). Moving on from counting strategies increases your ability to deal with larger numbers and to solve problems more effectively (Steffe, 1994; Moss & Case, 1999; Mulligan & Wright, 2000). Additive thinking will allow you to solve multiplication problems, but it is only with multiplicative thinking that you can fully understand decimals and fractions (Baturo, 2000). Multiplicative thinking also gives you more mathematical power, allowing you to think in groups rather than in a unitary fashion. (Clark & Kamii, 1996; Mulligan, Mitchelmore, Outhred & Russell, 1997). From experiences in arithmetic we hope that children will form structures which support generalisations about numbers and operations (Warren & English, 2000). Algebraic thinking thus provides another level of abstraction above the operation-based strategies.

Research in each of the areas listed in Figure One has resulted in frameworks which try to organise children’s responses in a way that makes them easy to characterise and provides information about the type of thinking children are exploring. Addition and subtraction, particularly of single-digit numbers, is the most researched of these areas (Carpenter & Moser, 1983). Findings in this area emphasise the need for conceptual understanding (place value) rather than rote learning (Carpenter, Franke, Jacobs, Fennema & Empson, 1998, Hiebert & Wearne, 1996, Peters, 1997), shifting the focus from the mastery of written algorithms to mental computation. This review does not cover early number development, focusing rather on the strategies of upper primary aged children. For this reason we begin with strategies for multi-digit addition and subtraction.

Figure 1.
Diagram of the basic relationships between increasingly sophisticated forms of thinking investigated in the strategies literature.



Multi-digit addition and subtraction

Many studies in this area (Fuson et al, 1997, Carpenter et al 1998, Hiebert & Wearne, 1996, Carpenter & Moser 1983, Cooper, Hierdsfield & Irons, 1996) are longitudinal in design and follow changes in children's thinking over time. There has been some cross-cultural work, comparing the strategies of children in different countries at a given point in time (McIntosh, Nohda, Reys & Reys, 1995, Reys & Yang, 1998, Anghileri, 2001). Experimental work on explicit teaching of strategies to children (Klein, Beishuizen & Treffers, 1998) and changing problem solving conditions (Torbeyns, Verschaffel & Ghesquiere, 2001) has been undertaken. Correlational studies consider the role of short term memory, affect, metacognition and problem solving preferences in strategy choice and problem solving accuracy (Heirdsfield, 2001, McIntosh, 1996).

An overview of the frameworks arising from this literature is provided in Table 1. This table is a comparison of four different research programmes. There are considerable similarities in the frameworks described, although the terminology varies. The framework from the Fuson et al (1997) study is an aggregation of the findings of four separate projects.

Table 1.
Mental Strategies for Multi-digit Addition and Subtraction

Cooper, Hierdsfield & Irons, 1996	Klein, Beishuizen & Treffers, 1998	Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema 1997	Carpenter, Fennema & Franke, 1993
Counting – on or back		Begin-with-one-number methods – counting on is the beginning of more sophisticated sequential strategies	Counting
Right to left separated place value -split numbers - operate on 1s, then 10s	1010 (decomposition strategy)	Decompose tens and ones methods – regroup then add/subtract	Combining 10s and 1s
Left to right separated place value - split numbers - operate on 10s, then 1s	1010 (decomposition strategy)	Decompose tens and ones methods – Add/subtract, then regroup	Combining 10s and 1s
Aggregation - keep one number whole, split the other number	N10 (sequential strategy)	Begin-with-one-number methods - add on 10s, then1s	Incremental strategies
Mixed - mixed of aggregation and place value strategies		Mixed methods	Incremental strategies
Wholistic - compensation - levelling	N10C A10 (sequential strategies)	Begin-with-one-number methods – overshoot and come back	Compensation

Although these strategies vary in sophistication, with counting as the most concrete, and compensation and wholistic strategies as the most abstract, this is not necessarily a progression. Klein et al (1998) and Beishuizen (1999) emphasise a dichotomy between decomposition methods, where the numbers are broken up into parts, and sequential methods, where one number is the start point and 'jumps' are made from that number. This reflects Fuson et al (1997)'s categories of 'begin-with-one-number-methods' and 'decompose-tens- and -ones methods'. The notion of 'progression' is further complicated by the effects of problem type on the choice of solution strategy. (Carpenter & Moser, 1983; Klein et al, 1998). Children rarely respond to a set of problems with only one type of strategy. Multiple strategies are used both across problems and within problems (Cooper et al, 1996, McIntosh, 1996) resulting in a 'mixed' category in most frameworks. Flexibility in strategy choice is seen as part of competence in this area (Heirdsfield, 2001, Klein et al 1998, Carpenter & Moser, 1983, Torbeyns et al, 2001). Torbeyns et al (2001) tested children under two conditions – one where they had a choice about which strategy to use, and one in which they were forced to use a particular strategy. Children performed more quickly and accurately when allowed to choose their own strategy. The way in which problems are presented to children also affects their choice of strategy, with written presentations increasing the number of responses which follow the format of the formal algorithm (McIntosh, Nohda, Reys & Reys,1995).

Further evidence that the multi-digit addition and subtraction strategies presented in Table 1 are not a progression is given by McIntosh (1996) who found that none of the strategy types were found to be used solely by the more able children. While there were patterns in the data, children in all ability groups accessed the full range of strategies.

It is important to note that the development of addition and subtraction of single-digit numbers which underpins the ability to solve multi-digit problems is thought to follow a progression (Carpenter & Moser, 1983). This progression is summarised in Figure 2, to provide a background to the findings described above.

Figure 2.
The development of single digit addition and subtraction strategies.

Addition	Subtraction
Direct modelling	Direct modelling
Counting all	Separating from Separating to Adding on Matching
Counting	Counting
Counting all	Counting down from
Counting on from first	Counting down to
Counting on from larger	Counting up from given
Recalled number facts	Recalled number facts
Derived facts	Derived facts
Recalled facts	Recalled facts

Note. Taken from Carpenter & Moser, 1983.

Multiplication and Division

Multiplicative thinking differs qualitatively from additive thinking (Clark & Kamii, 1996, Baturo, 1997). Multiplicative thinking is based on grouping – seeing groups as units which can be manipulated, rather than as consisting of single items. Mastery of multiplicative thinking is necessary for effective understanding of decimals and fractions (Baturo, 1997). Multiplication and division problems can be solved in a range of ways, however, and research in this area indicates that many children continue to use additive thinking to solve multiplication problems (Clark & Kamii, 1996, Mulligan & Mitchelmore, 1996). Table 2 compares the frameworks proposed by five research projects.

Table 2.
Mental strategies for multiplication and division

Mulligan & Mitchelmore, 1997	Anghileri, 1989	Carpenter, Fennema & Franke, 1993	Kouba, 1989	Mulligan & Wright, 2000
Direct counting	Direct modelling Unitary counting	Direct modelling -grouping -measurement - partitive	Direct representation Double counting (division)	Perceptual unitary counting
Rhythmic counting	Rhythmic counting			Perceptual counting in multiples
Skip counting	Number pattern	Counting strategies: Skip counting Addition strategies Trial and error	Transitional counting	Figurative composite units
Additive calculation	Use of addition facts *	Deriving from addition	Addition or subtraction	Repeated addition or subtraction
Multiplicative calculation	Use of multiplication facts*	Derived/known multiplication facts	Recalled number fact	Multiplication and division as operations

*Anghileri (1989) sees these strategies as separate paths towards multiplication, rather than leading from addition directly to multiplication.

Comparison of Table 2 with Figure 2 shows that the development of strategies for solving multiplication problems parallels that of the development of addition and subtraction. Multiplication is usually introduced later than addition and subtraction; therefore the more concrete strategies are seen in the age group under consideration here (9-11 years). Some authors argue that this means that multiplication problems should be introduced to younger children, who can find ways to solve them (Carpenter, Fennema & Franke, 1993, Mulligan and Mitchelmore, 1997, Anghileri, 1989). This serves to emphasise a key problem in the literature on strategies - alluded to in the discussion on addition and subtraction above – the role of problem type.

Word problems, with contexts familiar to children (such as sharing out lollies at a party), can be readily solved with a direct modelling or counting strategy. Problems such as the ‘fish task’ used by Clark & Kamii (1996) which use multiplicative thinking in the form of a ratio task produce different response types. Clark & Kamii’s task directly accesses multiplicative thinking, while the other studies look at responses to multiplication problems, which may or may not involve multiplicative thinking (Mulligan & Mitchelmore, 1997, Mulligan & Wright, 2000, Kouba, 1989, Anghileri, 1981, Mulligan & Mitchelmore, 1996, Carpenter et al 1993). Clark & Kamii (1996) found that multiplicative thinking emerged early, but developed slowly with only 49% of fifth graders demonstrating effective multiplicative thinking.

Responses to multiplication problems have been found to vary by an interaction of age of child, the size of the numbers, the language used and the semantic structure of the problem (Anghileri, 1989,Mulligan & Mitchelmore, 1997). However there is not a 'one strategy to one semantic structure' relationship as proposed by early work in this area (Fischbein, Deri, Nello & Merino, 1985). How children move from additive to multiplicative thinking is not made clear. Anghileri (1989) proposes that moving from unitary counting to rhythmic counting shows that children have recognised the composite nature of numbers (p. 376), and sees additive concepts feeding in to multiplicative concepts from then on. Mulligan & Mitchelmore (1997) suggest that teachers should "…develop …facility with addition and repeated addition at the same time". When children are competent with repeated addition, the idea of multiplicative operations should be introduced. Clark & Kamii (1996) recommend children should be allowed to solve multiplication problems '…in their own ways' (p. 50). While these authors suggest linking addition and multiplication in learning, not moving from additive to multiplicative thinking can have implications for understanding decimals and fractions (Baturo, 1997, Baturo 2000, Kouba, 1989).

Decimals and Fractions

The teaching of decimals in New Zealand schools begins at Level 3 of the Mathematics in the New Zealand Curriculum (Ministry of Education, 1993). Children of 9 to 11 years would be working at this level, beginning to work with decimals and extending their fraction understandings. Fractions and decimals are an extension of multiplicative thinking.

As suggested previously, continuing to think in certain ways leads to errors in higher-level tasks, such as understanding decimals. Treating rational numbers (fractions and decimals) as whole numbers leads to systematic errors in children's performance (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989, Mack, 1993). Children may learn isolated 'techniques' for dealing with fractions and decimals, but not make sense of these. Understandings about fractions and decimals then become disconnected from their place in the number system, within the child's mind (Mack, 1993, Reys & Yang, 1998). Evidence that children working on contextualised decimal problems improved their understanding more than children who worked on decontextualised problems (Irwin, 2001) suggests that connection and sense making are key elements in the development of strategies for decimal numbers.

The underlying conceptual framework used by children when they encounter decimals has a profound influence on their learning (Resnick et al, 1989, Mack, 1993, Baturo, 2000). Children who apply a 'fraction rule' to decimals understand them as parts of a whole, whereas children who apply a 'whole number' rule to decimals make errors such as identifying longer decimals as larger numbers (Resnick et al, 1989).

Place Value and Number Sense

Underlying children's strategies in approaching problems are their conceptions about number and their understanding of the number system. Place value and number sense are thus essential elements of children's strategic thinking. Three frameworks for considering a progression in thinking about the Base 10 number system are compared in Table 3.

Table 3.
Frameworks for describing place value understanding

Young-Loveridge, 2000	Fuson, Smith & LoCicero, 1997	Jones, et al 1996
Unitary concept	Unitary	Pre-place value
Ten-structured concept	Decade and ones	Initial place value
	Sequence tens and ones	Developing place value
Multi-unit concept	Separate tens and ones	Extended place value
Extended multi-unit concept		Essential place value

Fuson's (Fuson, Smith & Lo Cicero, 1997, Fuson & Smith, 1999) UDSSI triad model emphasises the transition between seeing a collection of items as separate objects and seeing them as a group which itself can be manipulated as a mathematical object. The UDSSI triad (Fuson et al, 1997) also expresses the links between the numerals, the words and the quantities in multi-digit numbers. 'Fifty-three' needs to be connected with 53 and with the amount fifty-three by the learner. An incorrect conception can be developed here. Children associate '53' with five and three, not realising that the '5' stands for 5 tens. This is an extension of unitary thinking, where a numeral stands for an amount, not for groups of an amount. This clearly relates back to the issues with additive and multiplicative thinking; the key is to think about groups as units. All three authors discuss the importance of coming to see a group of ten as a unit, and moving from unitary, counting-based understandings to a part-whole understanding of numbers as composites.

Children's Structures

Research from all the fields discussed can be drawn together in two ways. The first is to reiterate the general trend from the concrete to the abstract in children's thinking. Figure 3 summarises some of the trends described by the research which can be linked in this way.

Figure 3.
Trends in children's strategy development

From	To	Reference
Particular strategies - focused on problem given	General strategies - focused on generalities	Bills & Gray, 2001
Procedural	Deductive	Gray, 1991
Counting strategies	Aggregation and wholistic strategies	Cooper, Hierdsfield & Irons, 1996
Counting more	Using place value more	McIntosh, 1996
Unitary conceptions	Grouping in tens	Fuson, Smith & LoCicero, 1997

These trends exist across a group of children, but not necessarily within each individual child. The second link between the fields discussed is the difficulties children have at key points in the frameworks. Gray & Tall (1994) describe as 'proceptual encapsulation' the process of bundling together a group of understandings into a more sophisticated and powerful way of thinking. They identify two key points as encapsulating repeated counting as addition and encapsulating repeated addition as multiplication. This encapsulating process is described by other authors as the building of conceptual structures (Resnick et al, 1989; Warren & English, 2000) or intuitive models (Mulligan & Mitchelmore, 1997) into which children fit new understandings. This idea clearly has its origins in the work of Piaget and Vygotsky, discussed into the theoretical underpinnings section above. These structures have been shown to have important consequences for children's strategy choice and development (McIntosh, 1996, Mulligan & Mitchelmore, 1997; Mulligan, Mitchelmore, Outhred & Russell, 1997).Mulligan & Mitchelmore (1997) conclude that "the intuitive model employed to solve a particular problem…does not reflect any specific problem feature, but rather the mathematical structure that the student is able to impose on it" (p.327). A structure underlies strategy use and strategy choice. It determines how children see and understand numbers and operations on those numbers. Using this analysis, mental strategies and written recording become not ends in themselves, but clues to the underlying structure the child has constructed about how numbers work. These structures can be very helpful and support more abstract thinking or they can work against acquiring new understandings (Mulligan & Mitchelmore, 1997, Heirdsfield, 2001, McIntosh, 1996). That children's strategic thinking should be guided by a structured view of mathematics makes sense of commonalities of strategy type within the different areas of number, and accounts for systematic errors of understanding which commonly occur.

As well as determining error patterns, children's conceptual structures guide their selection of strategies in problem-solving situations (Gray, 1991). Children show a preference for certain strategy types, and Gray (1991) proposes that it is these preferences form a hierarchy within individuals that determines strategy selection.

A key concern with the notion of structures is the differences between the structures developed by high and low ability children. More competent children are able to abstract mathematical principles from their experiences in arithmetic (McIntosh, 1996, Gray & Tall, 1994) and build effective structures (Mulligan & Mitchelmore, 1997). Less able children either do not join together the separate pieces of mathematics that they are taught, or join them together in a way which promotes misunderstandings (Mulligan & Mitchelmore, 1997, Gray & Tall, 1994). For example "…in cases of low achievers, children were unable to visualise any structural characteristics and were so focused on their idiosyncratic interpretation of reality that the mathematical tasks and the concrete objects embedded in them had a completely different meaning to that intended" (Mulligan & Mitchelmore, 1997, p. 366). Evidence from Gray (1991) shows that children of differing abilities have characteristic strategy preferences, with more competent children using deductive strategies (based on their knowledge) and less competent children using procedural strategies. High performance students were the only ones with a 'complete structural schema' of multiplicativity in Baturo's (1997) study. Fuson & Smith (1999) find that "typically in our classrooms the top children invent a range of methods, middle children use quantities and then move to a written numerical method (often the traditional algorithms) that they can explain and understand, and lower children struggle to carry out correct methods using ten-structured quantities" (p.164). Gray and Tall (1994) perceive the mathematics experiences of high and low ability children to be qualitatively different, because of the structures they create. They propose that a Matthew Effect (Stanovich, 1986) then occurs. The effective structures built by competent children readily make sense of new mathematics. They are able to add to what they know. Ineffective structures prevent less competent children from adding new concepts. They are not able to make sense of new material. From this point onwards, the children have differential experiences of the mathematics curriculum (Gray & Tall, 1994).

The relationship between recording and thinking

Many of the studies of children's strategies consider only mental computation methods (Beishuizen, 1999, Cooper, Heirdsfield & Irons, 1996; Hierdsfield, 2001; Bills & Gray, 2001; Klein et al, 1998). Mental computation has become a focus for reform of mathematics education because it allows children to use their conceptual understanding in solving problems, rather than using a written algorithm. Mental methods are seen as reflecting children's thinking about the problem, rather than their recall of a learned procedure. In addition, much of the mathematics of daily life is done mentally. Developing skill in mental computation has become the key to some mathematics programmes for upper primary children, with written methods delayed until sufficient conceptual understanding is displayed through the use of mental strategies (Ministry of Education, 2001a, Beishuizen, 1999, Buys, 2001). Mental computation is generally defined by authors as computation which is carried out without external supports or recording, and which involves more than the recall of known facts (Cooper, Heirdsfield & Irons, 1996; Fuson & Smith, 1999). Proficient mental calculators are seen as accurate and flexible in their use of strategies (Heirdsfield, 2001, Bills & Gray, 2001). Bills & Gray (2001) look at mental strategies in a broader sense than accurate computation, characterising children's strategies as particular to the problem to be solved, generic and general. General strategies involve deducing general arithmetic principles from the problem given, and generic strategies involve a known solution method being applied. 'Particular' strategies are used in response to more difficult questions, and are more likely to yield incorrect answers. High achievers are significantly more likely to use non-particular mental strategies to solve problems. Again, the progression from the more concrete and context bound (particular strategies) to the abstract and mathematically powerful (general strategies) is apparent.

The relationship of mental computation to the other aspects of the number domain in which it is embedded is often not explored. McIntosh (1996) states that "the ability to compute mentally in flexible ways is both a component and an indicator of number sense" (p.260). This implies that mental computation and number sense are bound up together; understanding of place value and operations are also necessary components of mental computation. Cooper, Heirdsfield & Irons (1996) describe mental computation as "a component of addition and subtraction knowledge … (relating) … to the other components of this knowledge: The concepts of addition and subtraction, the basic facts, pen and paper algorithms and word problems" (p. 148), but do not describe these relationships further. It is noted by Fuson & Smith (1999), however, that "number sense … (is) … desirable, but ... (involves) … different processes than exact mental computation" (p.194), suggesting that mental computation is more than a component of number sense.

The relationship between children's thinking and the written recording of mathematics is an important one. Children presented with problems to solve orally chose different solution paths to those chosen when the problems were presented visually (McIntosh, Nohda, Reys & Reys, 1995). Written recording of representations of numbers or of operations may aid children in solving problems, which they cannot cope with mentally (Fuson & Smith, 1999). Analysis of children's representations of problem solution can provide useful information about their mental structures. Mulligan, Mitchelmore, Outhred & Russell (1997) found that "low achievers were more likely to produce poorly organised, pictorial and iconic representations that were lacking in structure. High achievers, who used dynamic imagery, presented their solutions in well-structured but often unconventional ways" (p.366). The recording methods taught to children may influence the way they build structures and conceive of numbers (Bills & Gray, 1999; Anghileri, 2001; Wright & Gould, 2000, Mulligan & Mitchelmore, 1996). This relationship is little explored in the literature. Written recording, in particular the vertical algorithm form, is largely regarded as harmful to the development of sound conceptual understanding (Anghileri, 2001). Beishuizen (1999) discusses the use of the empty number line as a device for recording children's thinking. This method allows children to record intermediate steps in problem solution and keep track of their strategy. Fuson & Smith (1999) focus on the establishment of a 'generative tens structure' in children's minds, and state that using written numbers helps children to think more clearly about the problems they are solving: "In classrooms in which all methods are based on understanding and no method is compulsory, mental computation no longer confers the advantage of more understanding. We certainly advocate self-developed strategies based on conceptual knowledge, but strategies do not have to be mental to meet this criterion" (p.194). Considering further the ways that children could use written recording to enhance their mathematical understanding is a worthwhile direction for future research.

Teaching and learning of strategies

Gathering information about children's strategic thinking in number provides direction for change in mathematics education and teaching methods. Mathematics is taught differently around the world, differing in both approach and content emphasis. Even within the group of programmes concerned with promoting the use of children's thinking in the classroom there are differences in materials, emphasis and methodology (Fuson & Smith, 1999). Several studies compare children in different countries (Anghileri, 2001, Reys & Yang, 1998, McIntosh et al 1995,) to see if there are differences in their approach to problems. These studies suggest that there are differences brought about by instructional focus. This is particularly clear in Anghileri's (2001) study, which compares children in Great Britain and The Netherlands solving division problems. Children in the Netherlands use a method, which arises from mental division strategies, and are more successful in solving division word problems than the British children. The model of the empty number line (Beishuizen, 1999) has been an important influence in The Netherlands, providing both a way of thinking about solving problems and a means of recording thoughts.

Other studies report on the results of teaching programmes which attempt to use knowledge about children's thinking to drive the mathematics curriculum (Fennema, Carpenter, Franke, Levi, Jacobs & Empson, 1996, Fuson et al 1997, Buys, 2001). Cognitively Guided Instruction (CGI) (Fennema et al, 1996) instructs teachers in children's strategic thinking in number. Reporting on a four-year longitudinal study with 21 teachers, Fennema et al (1996) conclude that "developing an understanding of children's mathematical thinking can be a productive basis for helping teachers to make the fundamental changes called for in current reform recommendations"(p. 403). The 'Children's Maths Worlds' project (Fuson et al., 1997; Fuson & Smith, 1999) focuses on using research in place value to build children's knowledge of the structure of numbers. Their results with low socio-economic students in the United States suggest that their programme produces considerable gains in children's understandings. Realistic Mathematics Education (RME) has inspired attempts to teach strategies (Klein, Beishuizen & Treffers, 1998) and the development of 'learning trajectories' which interpret the development of mathematical concepts for teachers (Buys, 2001). 'Count Me In Too' has been developed in Australia to reform curriculum in line with children's has been developed in Australia to reform curriculum in line with children's thinking (Wright & Gould, 2000). New Zealand's 'Early Number Project' (ENP) and 'Advanced Number Project' (ANP) are also based on this work (Ministry of Education, 2001a). An evaluation of the ENP pilot project found that "…teachers change their classroom practices to accommodate their new knowledge and understandings" (Thomas & Ward, 2001, p.ii), and that the children in these classrooms made gains beyond what would usually be expected of them. Teaching teachers about children's strategies and using teaching techniques which arise from a theoretical analysis of these strategies seems to result in teacher change and enhanced outcomes for children (Fennema et al, 1996, Fuson et al, 1997, Thomas & Ward, 2001).

Methodological Issues

Some common methodological issues arise from the research summarised here. Almost without exception this work is based on interviews with children. Carpenter & Moser (1983) conclude that "…individual interviews provide the most direct measure of the processes that children use" (p. 19), but note several limitations with this assessment method. In any format, probing for children's strategies is problematic because it involves the child's ability to describe their strategy, or the interviewer's ability to infer it. Children may use different strategies in an interview setting, believing they know what the interviewer wants.

Another result which characterises the research reported here is the variability in responses from children. Individual children used a wide range of strategies in response to problems (for example: Cooper, Heirdsfield & Irons, 1996). Some researchers attempt to code all the strategies used to solve a problem (McIntosh, 1996), others code the child's 'dominant' strategy only

(Mulligan & Mitchelmore, 1997). Beishuizen's (1999) distinction between strategies and procedures divides children's problem responses into parts. In other research these would be coded together. Coding of responses from children relies on recording of their behaviour and rater reliability. Information about the child's strategies may be lost if information is only audio-recorded, for example. In addition, some researchers discard data about unsuccessful strategies, when children do not manage to solve the problems. Insights into children's conceptual structures are lost in this way, as children's errors may also have a characteristic pattern.

Progress and product

If information about children's thinking in number is to inform classroom practice there needs to be a useful teaching sequence to follow. The goal of teaching must be known. There are two aspects to this in the literature. One is that strategies need to increase in sophistication, and the other is that thinking needs to be flexible. The initial parts of each of the proposed frameworks (for example, multiplication and division, Table 2) show a progression through strategies of increasing sophistication (modelling – counting – fact-based). Once this progression has been mastered, flexibility seems to be the key goal (for example, multi-digit addition, Table 1). Flexibility in strategy use would also seem to be strongly linked to number sense, as the choice of strategy is often dependent on the amounts involved, the operation to be performed and the way the question is posed (Carpenter & Moser, 1983; Kouba, 1989; van Lieshout, 1999; Verschaffel, 1999).

It is unclear from the literature what 'drives' this progression. While the work arises from a Piagetian tradition, researchers are not promoting an invariant, universal, developmental progression (Fuson et al., 1997; Bills & Gray, 2001). Some research has centred on what might help children to progress, with ideas about models and methods that might be useful (Fuson & Smith, 1999; Klein, Beishuizen & Treffers, 1998; Buys, 2001). Inter-country research suggests that the sequence is responsive to teaching, with children developing characteristic solution styles based on the models and methods used to teach them (Anghileri, 2001). Underlying children's development in number may be strong cognitive preferences. Gelman (1999) writes of naïve mathematics that "the favoured mathematical entities are the natural numbers; the favoured operations addition and subtraction, even if the task is stated as multiplication or division" (p. 576). This reflects the findings with school children, where the 'leap' to multiplicative thinking provides one of the key hurdles in children's progress. The relationship between the development of increasingly sophisticated thinking in number and children's general cognitive development has not been widely explored. The three key points of change seem to be from counting all strategies to using counting-on, from additive thinking to multiplicative thinking and from multiplicative thinking to proportional reasoning. These ways of thinking involve relationships between elements and processes that become increasingly complex, and may rest on more general principles of cognitive development.

An additional factor here is the children's own conceptual structures about the mathematical relationships in number. While the children are moving through the structure which researchers have developed as a descriptor of their progress, they are building their own unique structures and understandings which critically influence how they will continue to progress. The interplay between children's individual conceptual development and what we know about children's number learning in broader terms provides rich ground for research.

Conclusion

For children of ages 9 to 11 in New Zealand primary schools number learning focuses on place value, decimals and fractions, multi-digit addition and subtraction and multiplication and division. Research into children's thinking in these areas suggests a general progression from concrete thinking tied to physical models and counting methods, to abstract thinking using known number facts and relationships. To encourage children to 'make sense' of numbers and problems, mental calculation methods are promoted in the literature, with recommendations that written methods be delayed until conceptual understanding is established. Alternative models for recording, such as the empty number line or hundreds square are suggested to support the development of sound concepts in children. There appears to be a difference in strategy use between children who are successful in mathematics and those who are struggling. Children who are struggling with mathematics tend to continue to rely on counting strategies and additive thinking, while successful children use the more abstract and powerful ways of thinking.

Using information about children's thinking in teacher professional development has resulted in increased effectiveness in teaching. Although the suggested progressions are not developmental stages in a formal sense, they can guide instructional sequences and help teachers to decide 'where to next' for a child.

Looking beyond children's performance on measures of number skill is strongly suggested by this research. Each child is constructing a view of mathematics through their experiences which strongly influences the way they think and work with numbers. Knowing more about this will improve teaching and learning in mathematics classrooms

References

Anghileri, J. (1989). "An investigation of young children's understanding of multiplication." *Educational Studies in Mathematics, 20,* 367—385.

Anghileri, J. (2001). *What are we trying to achieve in teaching standard calculating procedures?* Paper presented to the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands, Freudenthal Institute.

Baturo, A. (1997). *The implication of multiplicative structure for students' understanding of decimal-number numeration.* Paper presented to the 20th Conference of the Mathematics Education Research Group of Australasia (MERGA), Rotorua, New Zealand.

Baturo, A. (2000). *Construction of a numeration model: A theoretical analysis*. Paper presented to the 23rd Conference of the Mathematics Education Research Group of Australasia (MERGA).

Beishuizen, M. (1999). Development of mathematical strategies and procedures up to 100. In M. Beishuizen (Ed.). *The role of contexts and models in the development of mathematical strategies and procedures* (pp127—161). Utrecht, The Netherlands: Freudenthal Institute.

Bills, C. and E. Gray (2001). *The 'particular', 'generic' and 'general' in young children's mental calculations.* Paper presented to the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands, Freudenthal Institute.

Buys, K. (2001). Mental arithmetic - example of a teaching/learning trajectory for multiplication. Utrecht, The Netherlands.

Carpenter, T. and J. Moser (1983). The acquisition of addition and subtraction concepts. In R. Lesh and M. Landau (Eds.). *Acquisition of Mathematics Concepts and Processes* (7–43). Orlando, FL: Academic Press.

Carpenter, T., Franke, M. et al. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3–20.

Clark, F. B. & Kamii, C. (1996). Identification of multiplicative thinking in children in Grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41–51.

Cooper, T., Heirdsfield, A., et al. (1996). Children's mental strategies for addition and subtraction word problems. In J. T. Mulligan and M. C. Mitchelmore (Eds.). *Children's Number Learning* (pp. 147–161). Adelaide, Aus.: The Australian Association of Mathematics Teachers.

Ministry of Education (1993). *Mathematics in The New Zealand Curriculum*. Wellington, NZ: Learning Media.

Ministry of Education (2001). *Advanced Number Project*. Wellington, NZ: Learning Media.

Fennema, E., Carpenter, T., et al. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*(4), 403–434.

Fischbein, E., Deri, M., et al. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education 16*, 3–17.

Fuson, K. C., Smith, S. T., et al. (1997). Supporting Latino First Graders' Ten-structured Thinking in Urban Classrooms. *Journal for Research in Mathematics Education, 28*(6), 738–766.

Fuson, K. C., Wearne, D., et al. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130–162.

Fuson, K. C. & Smith, S. T. (1999). Supporting multiple 2-digit conceptual structures and calculation methods in the classroom: issues of conceptual supports, instructional design and language. In M. Beishuizen (Ed.). *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 163–198). Utrecht, The Netherlands, Freudenthal Institute.

Gelman, R. (1999). Naive Mathematics. In R. A. Wilson and F. C. Keil (Eds.). *The MIT encyclopaedia of the cognitive sciences* (pp. 575–577). Cambridge MA: MIT Press.

Gray, E. (1991). An analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics, 22,* 551–574.

Gray, E. & Tall, D. O. (1994). Duality, ambiguity and flexibility: A 'proceptual' view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116–140.

Heirdsfield, A. (2001). *Integration, compensation and memory in mental addition and subtraction*. Paper presented to the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands, Freudenthal Institute.

Hiebert, J. & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and Instruction, 14*(3), 251–283.

Irwin, K. (2001). Using everyday knowledge of decimals to enhance understanding. *Journal for Research in Mathematics Education, 32*(4), 399–420.

Jones, G. A., Thornton, C. A.. et al. (1996). Multidigit number sense: A framework for instruction and assessment. *Journal for Research in Mathematics Education, 27*(3), 310–336.

Klein, A. S., Beishuizen, M., et al. (1998). The empty number line in Dutch second grades: Realistic vs gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443–464.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20*(2), 147–158.

Mack, N. (1993). Learning rational numbers with understanding: the case of informal knowledge. In T. Carpenter, E. Fennema & T. Romberg (Eds.). *Rational Numbers* (pp. 85–105). Hillsdale, NJ: LEA.

McIntosh, A., Nohda, N., et al. (1995). Mental computation performance in Australia, Japan and the United States. *Educational Studies in Mathematics, 29*, 237–258.

McIntosh , A., Reys, B. J., et al. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics, 12*(3): 2--8.

McIntosh , A. (1996). Mental computation and number sense of Western Australian students. In J. T. Mulligan & M. C. Mitchelmore (Eds.). *Children's Number Learning* (pp. 259–276). Adelaide, Aus: The Australian Association of Mathematics Teachers Inc.

Moss, J. & Case, R. (1999). Developing children's understanding of the rational numbers: a new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122–147.

Mulligan, J. T. & Mitchelmore, M. C. (1996). Children's representations of multiplication and division word problems. In J. T. Mulligan and M. C. Mitchelmore (Eds.). *Children's Number Learning* (pp. 163–183). Adelaide, Aus: The Australian Association of Mathematics Teachers Inc.

Mulligan, J. T. & Mitchelmore, M. C. (1997). Young Children's Intuitive Models of Multiplication and Division. *Journal for Research in Mathematics Education, 28*(3), 309–330.

Mulligan, J. T. & Mitchelmore, M. C., et al. (1997). *Second grader's representations and conceptual understanding of number: A longitudinal study*. Paper presented to the 20th Conference of the Mathematics Education Research Group of Australasia (MERGA), Rotorua, New Zealand.

Mulligan, J. T. & Wright, R. (2000). *Interview-based assessment of* *early multiplication and division*. Paper presented to the 24th Conference of the International Group for the Psychology of Mathematics Education, Hiroshima, Japan.

Peters, S. (1997). The relationship between place value understanding of seven-year-old children and the strategies they use to solve written addition problems. Paper presented to the 20th Conference of the Mathematics Education Research Group of Australasia (MERGA), Rotorua, New Zealand.

Piaget, J. (1952). *The child's conception of number*. New York: Norton.

Resnick, L. B., Nesher, P., et al. (1989). Conceptual bases of arithmetic errors; the case of decimal fractions. *Journal for Research in Mathematics Education, 20*(1), 8–27.

Reys, R. E. & Yang, D.-C. (1998). Relationship between computational performance and number sense among sixth and eighth grade students in Taiwan. *Journal for Research in Mathematics Education, 29*(2), 225–237.

Stanovich, K.E. (1986). Matthew Effect in reading: Some consequences of individual differences in the acquisition of literacy. *Reading Research Quarterly, 21,* 360–406.

Steffe, L., Cobb, P., et al. (1988). *Construction of Arithmetical Meanings and Strategies*. New York, Springer-Verlag.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel and J. Confrey (Eds.). *The development of multiplicative reasoning in the learning of mathematics* (pp. 3–40) Albany, NY: SUNY Press.

Thomas, G. & Ward, J. (2001). *An Evaluation of the Count me In Too Pilot Project*. Wellington, NZ: Learning Media.

Torbeyns, J., Verschaffel, L., et al. (2001). Investigating young children's strategy use and task performacne in the domain of simple addition, using the 'choice/no-choice' method. Paper presented to the 25th Conference of the International Group for the Psychology of Mathematics Education, Utrecht, The Netherlands, Freudenthal Institute.

van Lieshout, E. (1999). What can research on word and context problems tell about effective strategies to solve subtraction problems? In M. Beishuizen (Ed.). *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 79–111). Utrecht, The Netherlands: Freudenthal Institute.

Verschaffel, L. (1999). Young children's strategy choices for solving elementary arithmetic wor(l)d problems: the role of task and content variables. In M. Beishuizen (Ed.). *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 113–125). Utrecht, The Netherlands: Freudenthal Institute.

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Warren, E. & English, L. (2000). *Primary school children's knowledge of arithmetic structure*. Paper presented to the 23rd Conference of the Mathematics Education Research Group of Australasia (MERGA).

Wright, R. & Gould, P. (2000). *Reviewing literature relevant to a systemic early numeracy initiative: Bases of CMIT*. Paper presented to the 23rd Conference of the Mathematics Education Research Group of Australasia (MERGA).

Young-Loveridge, J. (2000). *How children's understanding of the number system varies as a function of ethnicity and socio-economic status*. Paper presented to the 23rd Conference of the Mathematics Education Research Group of Australasia (MERGA).